Solving for the Retirement Age in a Continuous-time Model with Endogenous Labor Supply*

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November 1, 2013

Abstract

This paper revisits a continuous-time life-cycle model with a consumption-leisure choice made by a finitely-lived agent with a random lifetime. We explicitly account for the leisure constraint in the corresponding constrained optimal control problem with a commonly postulated solution structure. By analytically solving both the constrained and unconstrained control formulations, we demonstrate the inaccuracy of the latter formulation (where an unconstrained leisure path gets truncated once it exceeds the time endowment limit) can indeed be significant. For cases when the subjective discount rate is quite close to or exceeds the interest rate, the optimal control path would not be consistent with the commonly postulated structure of the optimal solution.

JEL Classification: D91, C02, C61, J22, J26
Key Words: Constrained control; Pontryagin's Maximum Principle; Life-cycle consumption and labor-leisure

*We thank James Feigenbaum and Frank Caliendo for helpful comments. All errors are our own.
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1 Introduction

Finite lifetime, life-cycle models of consumption/saving with perfect foresight have become important building blocks for many quantitative-theoretical macro studies ever since Ando, Brumberg and Modigliani developed their ground-breaking works in the 1950s. Since then, numerous studies have enhanced our understanding of the microfoundations behind complex intertemporal consumption/saving decisions.[1]

Many such decisions fall into the realm of optimal control theory. Yet, mathematicians recognize that even simple optimal control problems (for which closed form solutions exist) often admit difficult analytic solutions. To remedy this, numerical optimal control is used.[2] Related economic models clearly constitute a subset of more general problems traditionally studied by control theorists in various fields. Such economic models are not easy to analytically solve. Consider, for example, a basic model with finitely-lived agents, including both intertemporal consumption and intertemporal labor/leisure choices.[3] While the form of utility function often allows one to ignore the consumption constraint, we know empirically that leisure constraint does bind. This leads to a technical difficulty, as properly constrained optimization methods require first order conditions be augmented to include the complementary slackness condition, that needs to be analyzed systematically. In view of such complexity, however, it may be tempting to simplify the problem as follows. A researcher would simply plot the unconstrained leisure profile and then look to see where the constraint binds. When the unconstrained leisure value is greater than one, the researcher replaces the unconstrained value with the number 1. To ensure this method "works", all that remains is to adjust the life-cycle budget constraint accordingly by finding the Lagrange multiplier (and thus the consumption/saving path) that would satisfy the budget constraint. Perhaps due to the existence of such temptations, some well-known mathematical economists have explicitly warned against solving an unconstrained control problem, and deleting the unfeasible portion when the latter binds (Kamien and Schwartz, 1981, pp. 83-87; p. 174).[4] The authors illustrate a production planning problem with inventory holding and production costs that shows simplifying things in the way described above clearly provides the wrong answer (see also Lenhart and Workman, 2007, pp. 79-80).

In this paper, therefore, we revisit the leisure constraint issue in a standard continuous-time model presented as an autonomous system with both endpoints on the state variable fixed, and use constrained optimization techniques of control theory to analyze the model. Such a conventional textbook problem has been considered in Bütler (2001). It should be noted that Bütler (2001) does not solve a corresponding bounded control problem, even though the possibility of an active leisure constraint is considered. The author carefully derives the growth rates of consumption and the consumption-leisure relationship when the leisure constraint happens to bind or not bind. Bütler's analysis enables us to understand the dynamics of the control variables, and is particularly useful for the cases when the leisure control does not bind (Bullard and Feigenbaum 2007).[5] We study the

[1] The literature on this is too large to properly cite in this paper. However, several notable studies can be mentioned. These include Yaari (1965), Hakansson (1969), Heckman (1974), Davies (1981), Skinner (1985), Zeldes (1989), Bütler (2001), Gourinchas and Parker (2002), Hansen and Imrohoroğlu (2008), Feigenbaum (2008), and Huang et al. (2012).

[2] For an excellent discussion of modern numerical techniques to tackle a large spectrum of optimal control problems, as well as associated difficulties and shortcomings, see, e.g., Gregory and Lin (1992), von Stryk (1993), Chachuat (2007), Ross (2009), and Betts (2010).

[3] Although the following list is not exhaustive, for variation of such models an interested reader may refer to İmrohoroğlu et al. (1995), Fuster et al. (2003), Heijdra and Ligthart (2002), Conesa and Garriga (2008), Bagchi (2011), and Park (2011).

[4] Intuitively, the reason why such a simplified (we call it "unconstrained truncation") approach is not a correct solution, is because the agent would properly anticipate the binding constraint and therefore may choose his leisure/consumption program in an altogether different manner that does not line up with the leisure/consumption program calculated in the above truncation manner. We thank Frank Caliendo for this clarification.

[5] Another important contribution is a line of research by Heckman and Macurdy (1980), Prescott et al. (2009), Rogerson and Wallenius (2009), and Rogerson (2011), who consider a continuous-time model with a finite lifetime and endogenous labor supply. Heckman and Macurdy (1980) characterize the optimal work schedule, yet they abstract from the evolution of the asset profile, and do not explicitly solve the model based on a systematic analysis of the complementary condition. Prescott et al. (2009), Rogerson and Wallenius (2009), and Rogerson (2011) focus on non-linear mapping from work hours to labor services, and intentionally

lifetime leisure, consumption, and capital profile evolutions with the general isoelastic, Cobb-Douglas utility function, and fully solve the model based on the complementary condition of bounded optimal control. Thus, our study complements the above contributions.

Below we consider an intertemporal utility-maximization problem of a finitely-lived agent with a random length of life, and first solve it based on a reasonably postulated structure of the optimal solution (where the agent's hours of work respond to changes in wage earnings over the life cycle, and the retirement decision is irreversible). We derive the analytic solutions to the optimal control, state and costate paths. Having an analytic solution for the leisure path is important as the effects of a parametric change on both the *timing* of the retirement ("optimal stoppage"), and the *intensive* margin of the labor can be clearly seen.[6]

After solving the constrained control problem, we reformulate the problem by intentionally disregarding the leisure constraint, and solve an unconstrained control problem. For such a problem, when the leisure constraint binds, an "unconstrained truncation" approach simply truncates the unfeasible portion of the leisure path and replaces it with the value 1. Then, all that remains is to adjust the budget constraint accordingly, so first-order conditions for the optimal consumption path are satisfied. We then resort to numerical demonstrations to compare these two approaches.

Our findings are as follows. First, for a large range of reasonable parameterizations where the subjective discount rate is close to or exceeds the interest rate, the postulated structure of the optimal solution fails to satisfy the state continuity condition. Some notable life-cycle studies with an exogenous labor supply suggest the discount rate is higher than the interest rate (e.g., Gourinchas and Parker 2002). Our model with endogenous labor supply, however, reveals the existence of the model's solution is highly sensitive to the gap between the time discount and interest rates, and this finding should be taken into account in future work. It may be the case that in order to analytically solve a life-cycle model with constrained leisure, one may have to assume an empirically unrealistic sequence of optimal arcs. Second, we find that for reasonably U and inverted-U shaped working life leisure and consumption paths, the correct formulation of the problem and the simplified, unconstrained truncation approach, produce optimal paths that are remarkably close to each other. Third, there exists a reasonable permutation of parameters, capable of delivering notably high gaps between the constrained and unconstrained truncation formulation's predictions. Such gaps can be large when it is optimal to have either a high or low retirement age.

We need to mention the reason we work in continuous time is because such life-cycle models are relatively scarce compared to discrete-time intertemporal models where quite a few authors seem to take the complementary slackness condition seriously (e.g., Auerbach et al. 1981). Further to this, certain economic phenomena (like fertility choice in an overlapping-generations framework) can be more conveniently captured in continuous time. In addition, Angeletos et al. (2001) and Caliendo (2011) cite instances regarding hyperbolic discounting that would ideally require high frequency analysis of intertemporal choices.

In what follows, Section 2 presents the solutions to both the constrained control and unconstrained control problem, and briefly considers a simplified truncation solution. Section 3 presents some numerical examples, and the last section provides the conclusion to this paper.

disregard the time-varying consumption and capital choices within a zero interest, a zero discount rate steady state, where the utility function is separable in consumption and leisure.

[6] In previous related continuous time studies with no such explicit solution, and to simplify the calculation burden, the authors assume constant (e.g., Caliendo and Gahramanov 2013) or endogenous yet still constant (Gahramanov and Tang 2013) intensive margins of labor supply, and then separately numerically calculate the optimal timing of retirement. (See also Bagchi (2011) on this.) Similarly, Samwick (1998) models leisure as a binary choice. Farhi and Panageas (2007) also assume the agent inelastically supplies leisure during the worklife, yet decides when to stop working. Koo et al. (2013) adopt a dynamic programming method to a variation of Merton's portfolio choice problem, and carefully derive closed-form solutions to the leisure path, however the authors consider an infinitely-lived economic agent. Similarly, Barucci and Marazzina (2008) study the Merton problem within an infinite horizon life setting. Zhang (2010) also considers an infinitely-lived agent with a stochastic labor income and asset return. For similar financial market settings refer also to Choi et al. (2008). Bagchi and Feigenbaum (2013a; 2013b) focus on a fixed planning horizon but their model is cast in discrete time.

2 Model: Basic Setup

Time is continuous and denoted by t. The representative agent enters the workforce at birth ($t = 0$). Let $Q(t)$ denote the probability of surviving until age t, which is a strictly positive and decreasing C^1 function. The individual definitely exits the model by age $t = T > 0$. If employed, the agent earns a market-determined constant wage, w, per labor efficiency unit, $\epsilon(t)$. The latter is generally not constant and varies over the life-cycle to mimic common empirical observations. All wage income not consumed flows into the individual financial asset account $k(t)$, which grows at the market rate of interest, r. The individual starts the life-cycle with no assets, and if he survives till age T, he finishes the life-cycle with no assets either. We follow Bütler (2001) and introduce mortality risk via the relevant discount factor in the agent's lifetime utility, and assume the government confiscates the assets of deceased agents to use those assets for purely consumptive purposes. Even though at any instant in time there may be some people who die with negative asset holdings, the government can offset overall debt via the asset holdings of those who die with positive savings.

We consider a standard intertemporal utility maximization by a representative agent. Preferences over consumption and leisure are given by the instantaneous utility function

$$U(c(t), l(t)) = \frac{(c(t)^{\phi} l(t)^{1-\phi})^{1-\sigma}}{1-\sigma}, \tag{1}$$

where $\sigma > 0$ (and $\sigma \neq 1$) is the inverse elasticity of intertemporal substitution, and $0 < \phi < 1$ captures the trade-off between consumption and leisure.[7] Time endowment of agents is normalized to unity, so $l(t)$ is defined as the fraction of time the agent devotes to non-work activity.[8]

2.1 Household Problem: Constrained Control

Let the rate of time preference be denoted by $\rho > 0$. The agent's problem can be formulated as

$$\underset{\{c(t), l(t)\}}{Max} \int_0^T Q(t) e^{-\rho t} U(c(t), l(t)) dt \tag{2}$$

subject to the trajectory (or state) equation, control region, and the end-point conditions, given in (3)-(6), respectively, as below:

$$\frac{dk(t)}{dt} = rk(t) + w\epsilon(t)(1 - l(t)) - c(t), \quad \text{for } t \in [0, T], \tag{3}$$

$$l(t) \leq 1, \tag{4}$$

$$k(0) = 0, \tag{5}$$

$$k(T) = 0. \tag{6}$$

Let us define the Hamiltonian function

$$H_1 = Q(t) e^{-\rho t} \frac{(c(t)^{\phi} l(t)^{1-\phi})^{1-\sigma}}{1-\sigma} + \mu(t)(rk(t) + w\epsilon(t)(1 - l(t)) - c(t)), \tag{7}$$

where $\mu(t)$ is a time-varying multiplier (costate or adjoint variable).[9]

[7] See, e.g., Bütler (2001) and Ladrón-De-Guevara et al. (1999) for details.

[8] Further, it can be shown the constraint $c(t) \geq 0$ would not bind, so our only focus hereafter would be on the leisure constraint.

[9] The analysis in this section heavily relies on the presentations in Kamien and Schwartz (1981), Gregory and Lin (1992), and Ross (2009).

Optimal controls must be chosen so the following conditions are satisfied:

$$\frac{dk(t)}{dt} = \frac{\partial H_1}{\partial \mu(t)} = rk(t) + w\epsilon(t)(1 - l(t)) - c(t), \tag{8}$$

$$\frac{d\mu(t)}{dt} = -\frac{\partial H_1}{\partial k(t)} = -\mu(t)r, \tag{9}$$

and

$$\max_{\{1-l(t)\geq 0, c(t)\}} H_1 \Leftrightarrow \max_{\{1-l(t)\geq 0, c(t)\}} \{H_2 = Q(t)e^{-\rho t}\frac{(c(t)^{\phi}l(t)^{1-\phi})^{1-\sigma}}{1-\sigma} + \mu(t)(w\epsilon(t)(1-l(t)) - c(t))\}. \tag{10}$$

Necessary condition is that there exists a time-dependent multiplier $\lambda(t) \leq 0$, so that if the Lagrangian of the Hamiltonian[10]

$$H = Q(t)e^{-\rho t}\frac{(c(t)^{\phi}l(t)^{1-\phi})^{1-\sigma}}{1-\sigma} + \mu(t)(w\epsilon(t)(1-l(t)) - c(t)) + \lambda(t)(l(t) - 1), \tag{11}$$

then the Lagrangian of the Hamiltonian is stationary with respect to the controls, and the complimentary slackness condition holds, implying

$$\frac{\partial H}{\partial c(t)} = Q(t)\phi e^{-\rho t}c(t)^{\phi(1-\sigma)-1}l(t)^{(1-\phi)(1-\sigma)} - \mu(t) = 0, \tag{12}$$

$$\frac{\partial H}{\partial l(t)} = Q(t)(1-\phi)e^{-\rho t}c(t)^{\phi(1-\sigma)}l(t)^{\phi(\sigma-1)-\sigma} - \mu(t)w\epsilon(t) + \lambda(t) = 0, \tag{13}$$

$$\lambda(t)(l(t) - 1) = 0, \tag{14}$$

$$1 - l(t) \geq 0. \tag{15}$$

Let us make a hypothesis about the structure of the solution. Since the age-productivity profile is hump-shaped (Feigenbaum 2008), it is reasonable to assume there may be some internal point in time (switching time) $t^* \in (0, T)$ (to be determined) on and after which the agent completely stops working, and hence optimal leisure is $l(t) = 1$. Such labor supply structure is commonly assumed in the literature as reasonable (e.g., Prescott et al. 2009; Rogerson and Wallenius 2009). Thus, we guess that

$$l(t) < 1 \text{ for } t \in [0, t^*), \tag{16}$$

$$l(t) = 1 \text{ for } t \in [t^*, T]. \tag{17}$$

Complementarity condition implies that if $\lambda(t) = 0$, then $l(t) < 1$, and we have the system of equations

$$\begin{pmatrix} k(t) \\ \mu(t) \end{pmatrix}' = \begin{pmatrix} rk(t) + w\epsilon(t)(1-l(t)) - c(t) \\ -\mu(t)r \end{pmatrix} \text{ for } t \in [0, t^*). \tag{18}$$

Similarly, if $\lambda(t) < 0$, then $l(t) = 1$, and we have the system of differential equations

$$\begin{pmatrix} k(t) \\ \mu(t) \end{pmatrix}' = \begin{pmatrix} rk(t) - c(t) \\ -\mu(t)r \end{pmatrix} \text{ for } t \in [t^*, T]. \tag{19}$$

[10] Note we require $\lambda(t) \leq 0$ here because we choose to adjoin $l(t) - 1$ to the Hamiltonian in (11), not $1 - l(t)$.

Hence, the solution to the problem can be found by piecing together the solution of (18)-(19).[11] In doing so, we first note the multiplier function $\mu(t)$ is defined over the entire region $t \in [0,T]$, and from (18) and (19) it clearly obeys the same law of motion on each subarc. Since the function is required to be continuous, we get

$$\mu(t) = ae^{-rt} \text{ for } t \in [0,T], \tag{20}$$

where a is a constant to be determined.

From (12) we deduce that

$$c(t) = ((1/(\phi Q(t)))e^{\rho t} l(t)^{(1-\sigma)(\phi-1)} \mu(t))^{\frac{1}{\phi(1-\sigma)-1}}. \tag{21}$$

Now, note that if $\lambda(t) = 0$, then $l(t) < 1$, meaning that (21), being substituted into (13), would result in

$$\mu(t) = ae^{-rt} = Q(t)(1-\phi)^{1+\phi(\sigma-1)} \phi^{\phi(1-\sigma)} e^{-\rho t} l(t)^{-\sigma} (1/(w\epsilon(t)))^{1+\phi(\sigma-1)}. \tag{22}$$

Recall that $l(t^*) = 1$. Using this in (22), we can express the constant a in terms of t^* as follows

$$a \equiv a(t^*) = Q(t^*)e^{(r-\rho)t^*}(1-\phi)^{1+\phi(\sigma-1)} \phi^{\phi(1-\sigma)} (1/(w\epsilon(t^*)))^{1+\phi(\sigma-1)}. \tag{23}$$

Thus,

$$\mu(t) = a(t^*)e^{-rt} \text{ for } t \in [0,T]. \tag{24}$$

Substituting (24) into (21) and recalling that $l(t) = 1$ if $\lambda(t) < 0$, we deduce from (19) that

$$\frac{dk(t)}{dt} = rk(t) - e^{\frac{\rho - r}{\phi(1-\sigma)-1}t}((1/(\phi Q(t)))a(t^*))^{\frac{1}{\phi(1-\sigma)-1}}, \tag{25}$$

for $t \in [t^*, T]$.

Using the boundary condition (6), we find the solution to (25) as

$$k(t) = e^{rt}(a(t^*)/\phi)^{\frac{1}{\phi(1-\sigma)-1}} \int_t^T ((1/Q(u))e^{(\phi r(\sigma-1)+\rho)u})^{\frac{1}{\phi(1-\sigma)-1}} du, \tag{26}$$

for $t \in [t^*, T]$. Here u is a dummy variable of integration.

Evaluating (26) at $t = t^*$, we obtain

$$k(t^*) \equiv \Lambda_1(t^*) = e^{rt^*}(a(t^*)/\phi)^{\frac{1}{\phi(1-\sigma)-1}} \int_{t^*}^T ((1/Q(t))e^{(\phi r(\sigma-1)+\rho)t})^{\frac{1}{\phi(1-\sigma)-1}} dt. \tag{27}$$

Next, substituting (24) into (12) and (13) and considering $\lambda(t) = 0$ case, we solve for the time-dependent consumption and leisure paths as functions of t^* (which is yet to be determined) as follows

$$c(t) = a(t^*)^{-\frac{1}{\sigma}} \Omega_1(t), \tag{28}$$

$$l(t) = a(t^*)^{-\frac{1}{\sigma}} \Omega_2(t), \tag{29}$$

for $t \in [0, t^*)$. Here

$$\Omega_1(t) \equiv Q(t)^{\frac{1}{\sigma}}(1-\phi)^{\frac{(\sigma-1)(\phi-1)}{\sigma}} \phi^{\frac{\sigma+\phi(1-\sigma)}{\sigma}} w^{\frac{(\sigma-1)(1-\phi)}{\sigma}} \epsilon(t)^{\frac{(\sigma-1)(1-\phi)}{\sigma}} e^{\frac{r-\rho}{\sigma}t}, \tag{30}$$

[11] For more details, see, e.g., Gregory and Lin (1992, pp. 12-13, 84-94), and Ross (2009, pp. 37-38).

$$\Omega_2(t) \equiv Q(t)^{\frac{1}{\sigma}}(1-\phi)^{\frac{1+(\sigma-1)\phi}{\sigma}}\phi^{\frac{\phi(1-\sigma)}{\sigma}}w^{\frac{\phi(1-\sigma)-1}{\sigma}}\epsilon(t)^{\frac{\phi(1-\sigma)-1}{\sigma}}e^{\frac{r-\rho}{\sigma}t}. \tag{31}$$

Substituting (28) and (29) into (18), we obtain

$$\frac{dk(t)}{dt} = rk(t) + w\epsilon(t)(1 - a(t^*)^{-\frac{1}{\sigma}}\Omega_2(t)) - a(t^*)^{-\frac{1}{\sigma}}\Omega_1(t), \tag{32}$$

for $t \in [0, t^*)$.

Using (5), we solve (32) as

$$k(t) = e^{rt}\int_0^t (w\epsilon(u)(1 - a(t^*)^{-\frac{1}{\sigma}}\Omega_2(u)) - a(t^*)^{-\frac{1}{\sigma}}\Omega_1(u))e^{-ru}du, \tag{33}$$

for $t \in [0, t^*)$.

Because of the required continuity of $k(t)$, we obtain from (33)

$$k(t^*) \equiv \Lambda_2(t^*) = e^{rt^*}\int_0^{t^*} (w\epsilon(t)(1 - a(t^*)^{-\frac{1}{\sigma}}\Omega_2(t)) - a(t^*)^{-\frac{1}{\sigma}}\Omega_1(t))e^{-rt}dt. \tag{34}$$

Hence, t^* is the solution to the following equation

$$\Lambda_1(t^*) = \Lambda_2(t^*). \tag{35}$$

Let "RHS" stand for "the right-hand-side" expression. We then summarize the solution to problem (2)-(6) for this section as

$$k(t) = \begin{cases} \text{RHS of Eq. (33) for } t \in [0, t^*) \\ \text{RHS of Eq. (26) for } t \in [t^*, T] \end{cases}, \tag{36}$$

$$c(t) = \begin{cases} \text{RHS of Eq. (28) for } t \in [0, t^*) \\ e^{\frac{\rho - r}{\phi(1-\sigma)-1}t}((1/(\phi Q(t)))a(t^*))^{\frac{1}{\phi(1-\sigma)-1}} \text{ for } t \in [t^*, T] \end{cases}, \tag{37}$$

$$l(t) = \begin{cases} \text{RHS of Eq. (29) for } t \in [0, t^*) \\ 1 \text{ for } t \in [t^*, T] \end{cases}, \tag{38}$$

where t^* solves (35), the costate variable is determined from (24), and $\lambda(t)$ is found from (13).

2.2 Unconstrained Control

Let subscript "un" denote "unconstrained". We need to solve (2) subject to (3), (5) and (6) (with the proper "un" subscripts wherever relevant), i.e., now the leisure constraint is disregarded. Pontryagin's Maximum Principle would imply the Hamiltonian given by (7), and the first-order conditions given by (8)-(9), (12) and

$$Q(t)(1-\phi)e^{-\rho t}c_{un}(t)^{\phi(1-\sigma)}l_{un}(t)^{\phi(\sigma-1)-\sigma} - \mu_{un}(t)w\epsilon(t) = 0. \tag{39}$$

The costate variable $\mu_{un}(t)$ has the general solution

$$\mu_{un}(t) = be^{-rt}, \tag{40}$$

for $t \in [0, T]$, and where b is a constant to be determined.

Proceeding in the manner similar to the scenario in Section 2.1, when $\lambda(t)$ was zero (see (22)), we get

$$\mu_{un}(t) = be^{-rt} = Q(t)(1-\phi)^{1+\phi(\sigma-1)}\phi^{\phi(1-\sigma)}e^{-\rho t}l_{un}(t)^{-\sigma}(1/(w\epsilon(t)))^{1+\phi(\sigma-1)}, \tag{41}$$

or,

$$l_{un}(t) = b^{-\frac{1}{\sigma}} Q(t)^{\frac{1}{\sigma}} e^{\frac{r-\rho}{\sigma}t} (1-\phi)^{\frac{1+(\sigma-1)\phi}{\sigma}} \phi^{\frac{\phi(1-\sigma)}{\sigma}} (1/(w\epsilon(t)))^{\frac{1+\phi(\sigma-1)}{\sigma}}, \tag{42}$$

for $t \in [0, T]$. Expression (42) can be stated more compactly as

$$l_{un}(t) = b^{-\frac{1}{\sigma}} \Omega_2(t), \tag{43}$$

for $t \in [0, T]$.

Thus, referring to (21), (40) and (42), we obtain

$$c_{un}(t) = b^{-\frac{1}{\sigma}} \Omega_1(t), \tag{44}$$

for $t \in [0, T]$.

Using (43) and (44) in (3), and applying the boundary conditions (5) and (6), we obtain that

$$k_{un}(T) = e^{rT} \int_0^T (w\epsilon(t)(1 - b^{-\frac{1}{\sigma}} \Omega_2(t)) - b^{-\frac{1}{\sigma}} \Omega_1(t)) e^{-rt} dt = 0. \tag{45}$$

The latter implies that

$$b^{-\frac{1}{\sigma}} = \frac{\displaystyle\int_0^T e^{-rt} w\epsilon(t) dt}{\displaystyle\int_0^T e^{-rt} (w\epsilon(t)\Omega_2(t) + \Omega_1(t)) dt}. \tag{46}$$

The optimal state path for the unconstrained control problem is given by

$$k_{un}(t) = e^{rt} \int_0^t (w\epsilon(u)(1 - l_{un}(u)) - c_{un}(u)) e^{-ru} du, \tag{47}$$

for $t \in [0, T]$.

2.2.1 An "Unconstrained Truncation" Approach to the Household Optimization

Assume the leisure constraint is tight. Thus, let there be an instant in time $t_{unt} \in (0, T)$ prior to which the unconstrained leisure path is less than 1 but after which the leisure exceeds 1. (Subscript "unt" denotes "unconstrained truncation".) Let us set all the unconstrained leisure values which are 1 (or higher) equal to number 1. Hence, a truncated leisure path is as follows

$$l_{unt}(t) = \begin{cases} \text{RHS of Eq. (43) for } t \in [0, t_{unt}) \\ 1 \text{ for } t \in [t_{unt}, T] \end{cases}. \tag{48}$$

Given $l_{unt}(t)$, in essence, we construct a new Hamiltonian with the costate variable $\mu_{unt}(t)$, so the Hamiltonian gets maximized only with respect to the consumption path $c_{unt}(t)$, given the trajectory equation for the assets holding, and the boundary conditions. In other words, we are choosing the costate variable $\mu_{unt}(t)$ to satisfy all the constraints, given that consumption is smoothly adjusted over the whole life-cycle based on the necessary conditions. Therefore,

$$\mu_{unt}(t) = q e^{-rt}, \tag{49}$$

for $t \in [0, T]$, where q is a constant to be determined. Necessary conditions would imply

$$c_{unt}(t) = ((1/(\phi Q(t))) e^{\rho t} l_{unt}(t)^{(1-\sigma)(\phi-1)} \mu_{unt}(t))^{\frac{1}{\phi(1-\sigma)-1}}. \tag{50}$$

Taking (3) into account, we must require

$$\frac{dk_{unt}(t)}{dt} = rk_{unt}(t) + w\epsilon(t)(1 - b^{-\frac{1}{\sigma}}\Omega_2(t)) - ((1/(\phi Q(t)))e^{\rho t}(b^{-\frac{1}{\sigma}}\Omega_2(t))^{(1-\sigma)(\phi-1)}\mu_{unt}(t))^{\frac{1}{\phi(1-\sigma)-1}}, \tag{51}$$

for $t \in [0, t_{unt})$,

$$\frac{dk_{unt}(t)}{dt} = rk_{unt}(t) - ((1/(\phi Q(t)))e^{\rho t}\mu_{unt}(t))^{\frac{1}{\phi(1-\sigma)-1}}, \tag{52}$$

for $t \in [t_{unt}, T]$, and

$$k_{unt}(0) = 0, \tag{53}$$

$$k_{unt}(T) = 0. \tag{54}$$

Using (53), the particular solution to (51) is

$$k_{unt}(t) = e^{rt}\int_0^t \left[w\epsilon(u)(1 - b^{-\frac{1}{\sigma}}\Omega_2(u)) - ((1/(\phi Q(u)))e^{\rho u}(b^{-\frac{1}{\sigma}}\Omega_2(u))^{(1-\sigma)(\phi-1)}\mu_{unt}(u))^{\frac{1}{\phi(1-\sigma)-1}}\right]e^{-ru}du, \tag{55}$$

for $t \in [0, t_{unt})$.

Using (54), the particular solution to (52) is

$$k_{unt}(t) = e^{rt}\int_t^T ((1/(\phi Q(u)))e^{\rho u}\mu_{unt}(u))^{\frac{1}{\phi(1-\sigma)-1}}e^{-ru}du, \tag{56}$$

for $t \in [t_{unt}, T]$.

Using (49), and matching (55) and (56) at $t = t_{unt}$, we solve for the constant q as follows

$$q = \left[\frac{\int_0^{t_{unt}} w\epsilon(t)(1 - b^{-\frac{1}{\sigma}}\Omega_2(t))e^{-rt}dt}{z}\right]^{\phi(1-\sigma)-1}, \tag{57}$$

$$\begin{aligned} z \equiv & \int_0^{t_{unt}} e^{-rt}[(1/(\phi Q(t)))e^{(\rho-r)t}(b^{-\frac{1}{\sigma}}\Omega_2(t))^{(1-\sigma)(\phi-1)}]^{\frac{1}{\phi(1-\sigma)-1}}dt \\ & + \int_{t_{unt}}^T e^{-rt}[(1/(\phi Q(t)))e^{(\rho-r)t}]^{\frac{1}{\phi(1-\sigma)-1}}dt. \end{aligned} \tag{58}$$

Thus, the costate variable $\mu_{unt}(t)$ which is needed to adjust the lifecycle budget constraint for a given truncated leisure path $l_{unt}(t)$, is determined via (49) and (57). The associated optimal consumption path is

$$c_{unt}(t) = \begin{cases} [q(1/(\phi Q(t)))e^{(\rho-r)t}(b^{-\frac{1}{\sigma}}\Omega_2(t))^{(1-\sigma)(\phi-1)}]^{\frac{1}{\phi(1-\sigma)-1}} \text{ for } t \in [0, t_{unt}) \\ [q(1/(\phi Q(t)))e^{(\rho-r)t}]^{\frac{1}{\phi(1-\sigma)-1}} \text{ for } t \in [t_{unt}, T] \end{cases}. \tag{59}$$

2.3 A Summary of the Solution to the Household Problem

We can now combine the results of Sections 2.1 and 2.2 to more formally specify the generalized solution to the consumer problem (2)-(6) as follows. If there exists $t^* \in (0, T)$ which solves (35), and the associated multiplier function $\lambda(t)$ satisfies the complementary slackness condition, then the solution to the problem (2)-(6) is given by (36)-(38). If such $t^* \in (0, T)$ does not exist, then either the constraint is inactive (and thus the solutions follow those in Section 2.2), or there are multiple switching times.

3 Simulation

We normalize the wage rate to $w = 1$, and assume the maximum life length is 100 years. As we model agents from age 25 onward, we set $T = 75$. We set our survival probability $Q(t)$ to Feigenbaum's (2008) sextic polynomial. Up to age 80 we also make use of Feigenbaum's quartic polynomial for the efficiency profile. Yet, since that polynomial starts picking up after about age 80, we assume that from $t = 55$ onward, the efficiency function remains continuous but steadily decays as a simple quadratic function, nearing zero by $t = 75$. The data, generated by piecing together both the Feigenbaum's polynomial and our quadratic function, are then used for curve fitting. This results in the following well-behaved quartic polynomial:

$$\begin{aligned}\epsilon(t) &= 0.888433666571627 + 0.04884518051585658t - 0.001648322482796714t^2 \\ &\quad +0.00001983885506181984t^3 - 1.13649956512419 \times 10^{-7}t^4,\end{aligned} \tag{60}$$

for $t \in [0, T]$. We can then choose r, ϕ, ρ, and σ to hit the following four targets: 1) consumption hump occurs at age 50; 2) the ratio of peak consumption to the initial consumption is about 1.1; 3) individual retires at age 65; and 4). the individual works 34 hours per week on average, before completely exiting the workforce. These targets are well supported in the literature (e.g., Feigenbaum 2008; Bagchi 2011).

Remark 1. If we define a loss function as squared percentage differences between the above targets (1-4) and the actual simulated results, we find under a wide range of feasible and reasonable parameterizations, that loss functions with lower values tend to produce insignificant gaps between solutions generated by the constrained control and the unconstrained truncated approach.

This suggests that for sufficiently U and inverted-U shaped leisure and consumption paths, respectively (with realistic work hours and consumption features), one would get essentially the same result under both problem formulations. However, some interesting results follow using a number of parameterizations, as explored below.

Remark 2. Considering quite a wide permutation of the four key parameters (r, ϕ, ρ, and σ), we observe that when the discount rate ρ exceeds the rate of interest r (and even when it is quite close to the interest rate), the constrained optimal control problem does *not* have the optimal solution of a postulated structure.

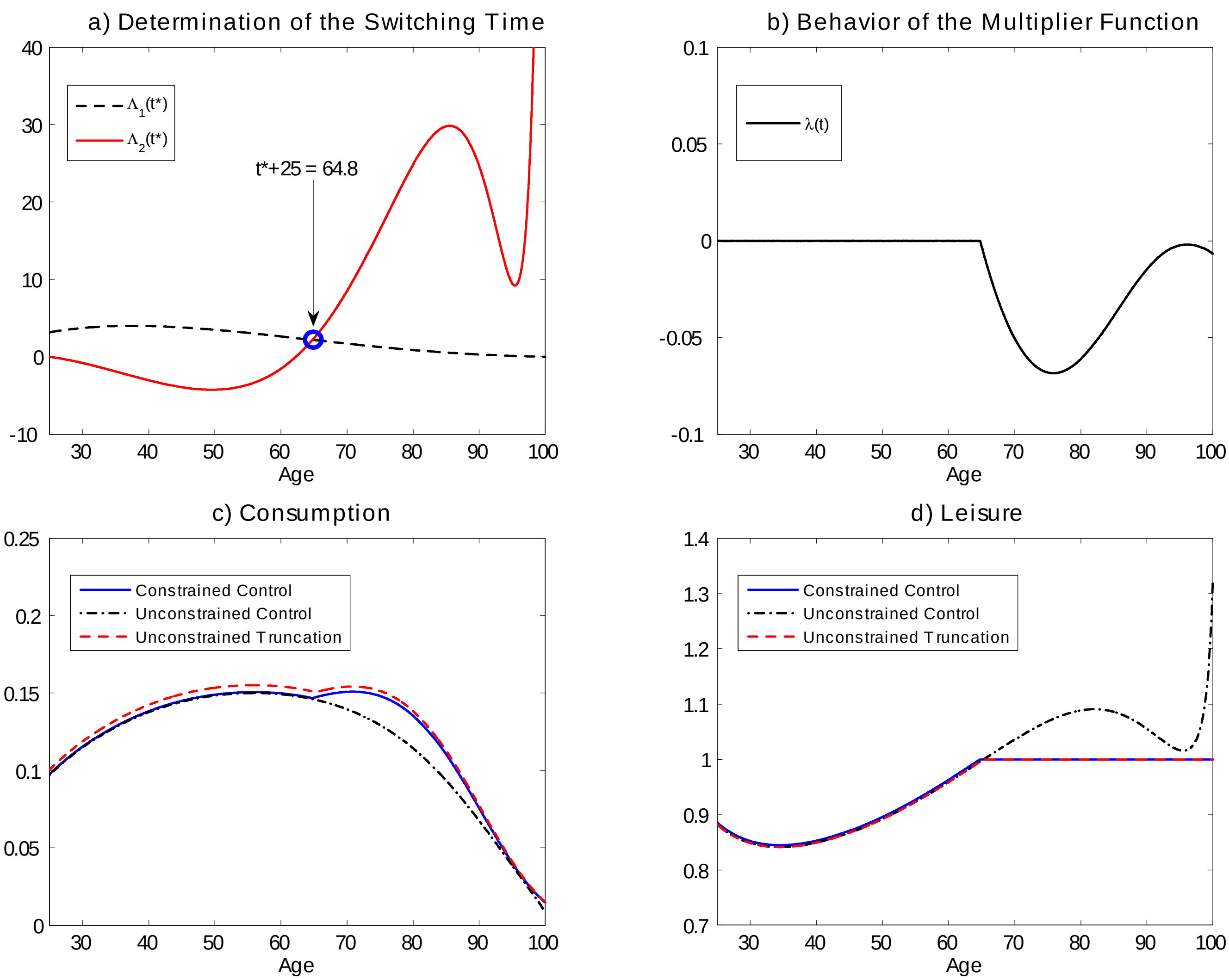
This is an important finding as it is not uncommon in quantitative-theoretical models to assume the discount rate of a typical agent exceeds the rate of return on private investment. Our results show the proposed optimal solution structure, so commonly assumed in the literature (where an agent works during the earlier years of his lifetime before he permanently leaves the workforce during more senior years), cannot ensure that state $k(t)$ varies continuously. In other words, functions $\Lambda_1(t^*)$ and $\Lambda_2(t^*)$ (see Eqs. (27) and (34)) simply do not cross for any $t \in [0, T]$. One may conjecture that when the intrinsic discount rate is higher than the interest rate, the consumption should start at a higher value, and then grow at a negative rate since the growth rate is proportional to $r - \rho + \frac{d \ln Q(t)}{dt}$ (thus, the growth rate of consumption will only become more negative as mortality risk increases). Since consumption and leisure are substitutes in the utility function, this would imply lower, but rapidly growing leisure values during younger years, with the leisure constraint starting to bind too early in the lifetime. After some period of idleness (where $l(t) = 1$), the agent may find it optimal to return to work to counter declining efficiency profile, and then retire permanently relatively closer to the end of the life-cycle. Hence, there may be multiple switching times. In addition, even if higher discount rates fall slightly short of the interest rate, consumption tends to start at a higher level during the early years, while leisure tends to start at a lower level, thus making it unlikely leisure would hit the constraint during later years. In a few instances, we observe the leisure constraint as simply inactive.

Remark 3. There are a number of feasible and reasonable parameterizations, where differences between the solutions for the constrained and unconstrained truncation approaches are not trivial. For a given path of the intensive margin of labor supply, such differences are quite sensitive to time when the agent decides to permanently retire in the constrained control problem. One can also find parameterizations where a notable gap between correct and incorrect solutions exist for both the relatively low and high retirement ages.[12]

Below we explore three particular cases.

Case 1. Let, for the sake of argument, $\sigma = 3.7$, $\rho = 2\%$, $r = 4.5\%$, and $\phi = 0.11$. Figure 1 plots the control and key dynamic variables associated with various problem formulations.

Figure 1

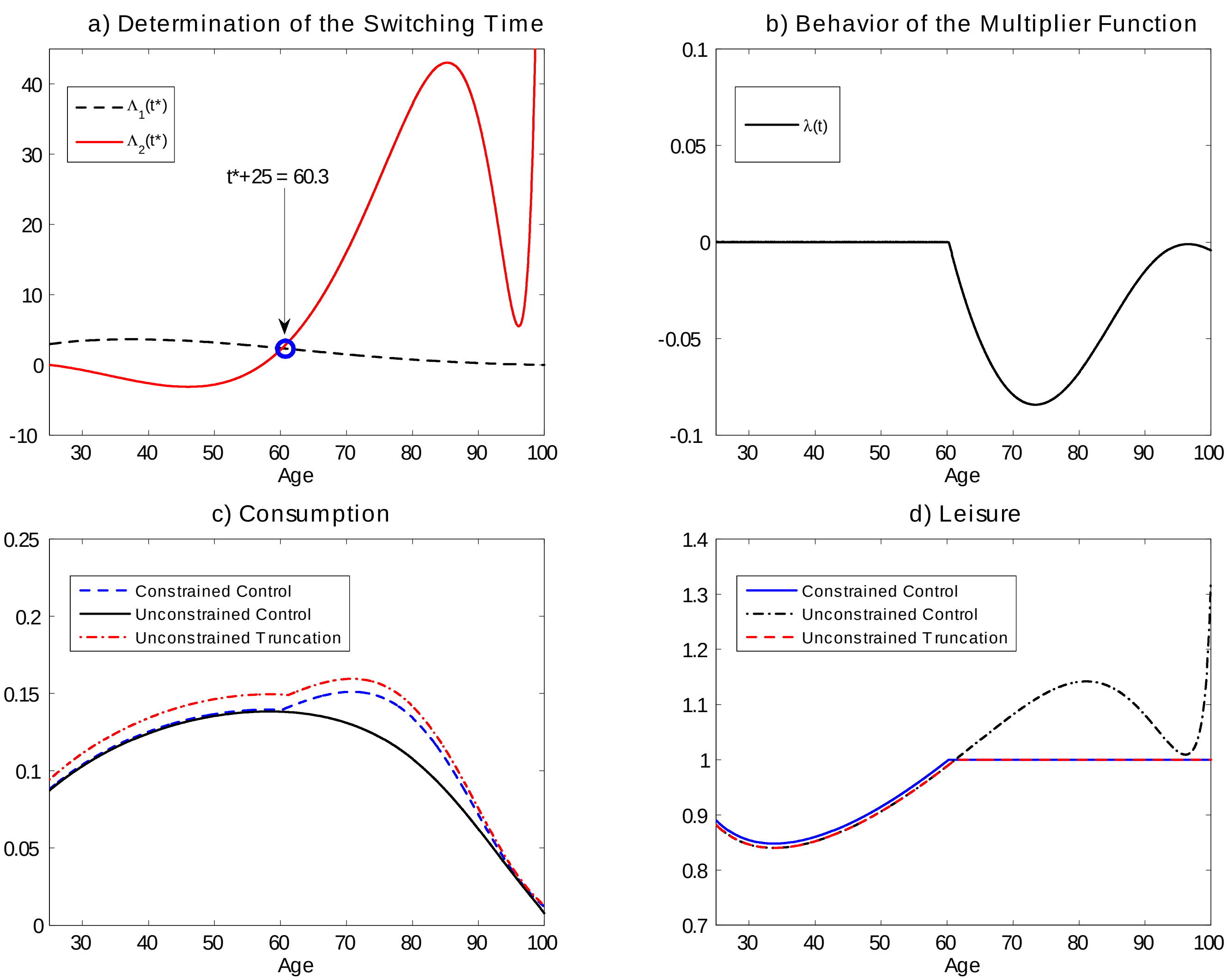


[12] We note that even though there is always one average cohort retirement age, in reality there is a noticeable heterogeneity in retirement timing both within and across countries (see, e.g., Gruber and Wise 2002; Cremer et al. 2004; Gustman and Steinmeier 2005). For example, Cremer et al. (2004) note that there is a trend towards early retirement in most European countries, while Gustman and Steinmeier (2005) find those with high discount rates often retire at age 62. Further, it is not unreasonable to believe that poorer individuals would tend to retire later in life.

Therefore, during the early years, the agent from the constrained control problem would retire at age 64.8. The peak consumption occurs at age 70.9, and the size of the consumption peak is about 1.5 (although, there is an earlier consumption hump of roughly the same magnitude occurring at around age 56, which is close to the far end commonly observed in the literature). The constrained control and unconstrained truncated solutions for leisure paths fall right on top of each other, yet we find an agent from the former problem would retire roughly half a year before the agent from the latter problem. The lifetime consumption profile from the constrained problem is, on average, less than that from the unconstrained truncated problem by about 2.6%.[13]

Case 2. Now let $\sigma = 3$, $\rho = 2\%$, $r = 4.5\%$, and $\phi = 0.1$. The associated solutions are plotted in Figure 2.

Figure 2



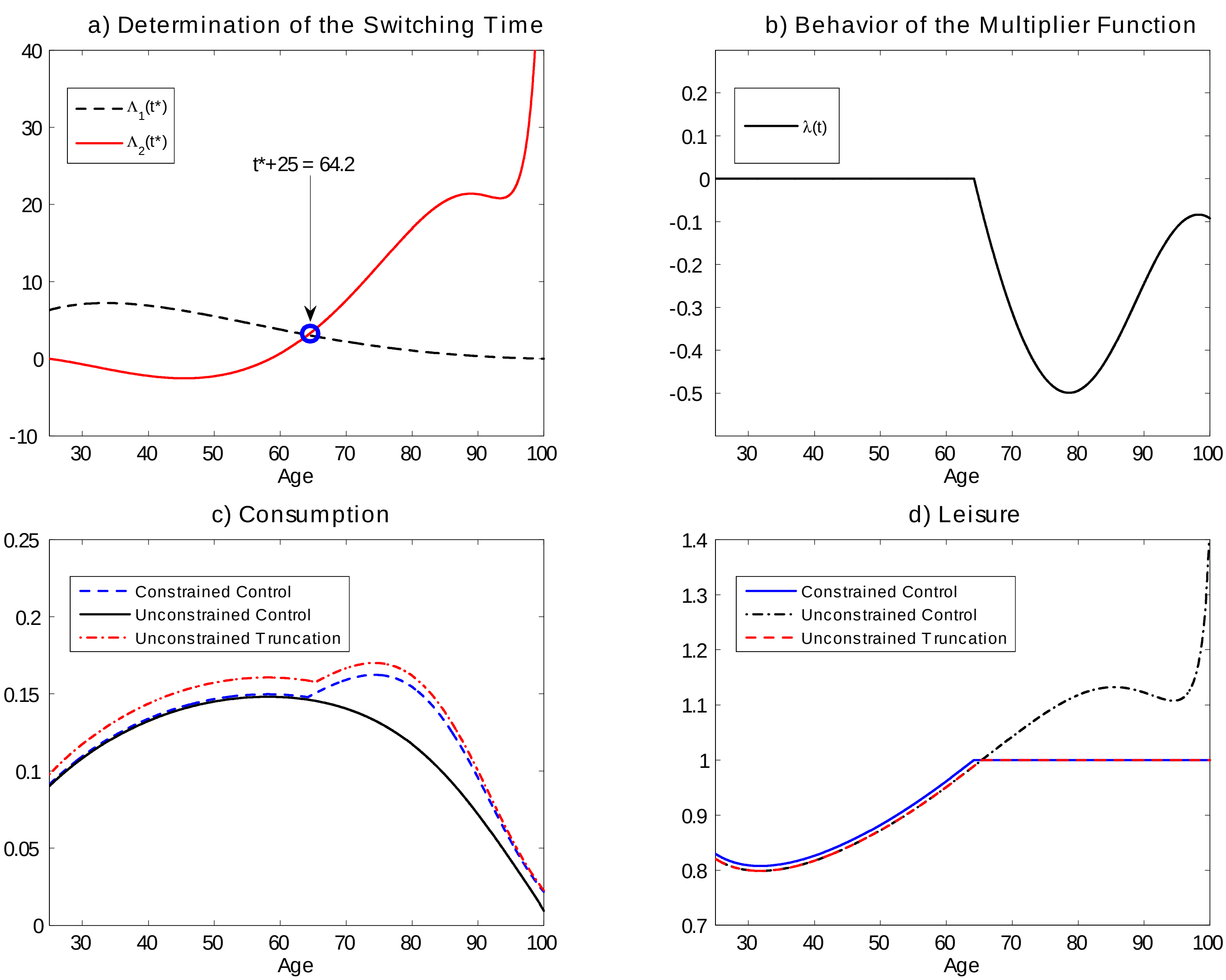
[13] We place the magnitude of these numbers into context by noting that elimination of idiosyncratic shocks to wage income, which is a major source of macroeconomic research, is equivalent (in terms of welfare gain) to a 1-2 percentage points increase in per period of consumption (Vidangos 2009). Furthermore, according to the estimates by Gruber and Wise (2005), raising the age of retirement eligibility in the United States by one year would lead to about a 10 percent savings to the system.

Comparing Case 1 and Case 2, we observe the dynamics of the constrained control leisure paths are similar, yet under current parameterization, the correct retirement age is much lower (about 60.3), while the unconstrained truncated setting predicts the retirement age of about 61.3, thus making the gap half a year wider than it was in Case 1 (see Fig. 1d and Fig. 2d). We also see the relative lifetime gap between the consumption paths of constrained control and unconstrained truncated settings significantly larger and equaling about 6.3%.

Next, although it may be tempting to think that a low value for retirement age implies that the leisure constraint binds more strongly (leading to a larger gap in the solutions), we observe that a notable gap between the correct and the truncated control solution may arise even if the retirement date is relatively high. This is shown in Case 3 below.

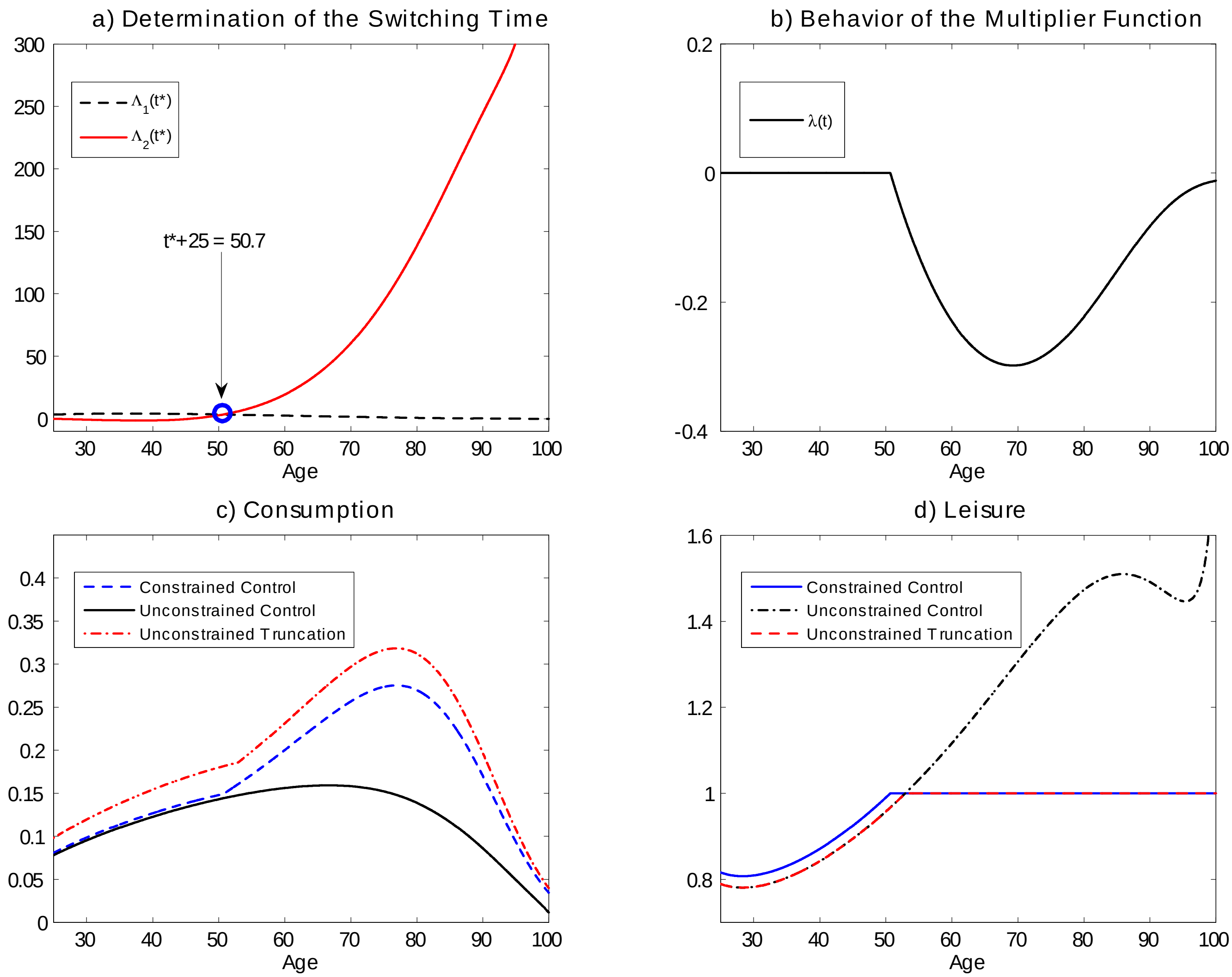
Case 3. Let $\sigma = 4.3$, $\rho = -1\%$, $r = 2.5\%$, and $\phi = 0.11$. We observe the optimal switching time in the constrained control setting isn't so small, and in fact equals 64.2. However, the unconstrained truncation formulation predicts that the optimal retirement age is 65.4, i.e., the gap between retirement age predictions exceeds one year. Further, the average relative gap between consumption paths predicted by both optimal control settings is also large and equals 6.1%.

Figure 3



We note that other parameterizations do exist when gaps in the leisure solutions are much bigger. If, for example, $\sigma = 3$, $\rho = 0.01$, $\phi = 0.1$, $r = 0.055$, the gap in the retirement dates is about 2.3 years, while the relative gap in the consumption paths can be as high as 17.8 percent. The results are shown in Figure 4.

Figure 4



4 Conclusion

We revisit a continuous-time, finite lifetime household maximization problem with mortality risk, where the optimal controls are consumption and leisure. We provide complete analytic solutions to the optimal control, state, and co-state paths with a properly constrained leisure path. We also solve two more formulations of the problem; one where the leisure constraint is completely ignored, and the other where the unconstrained value of leisure is replaced with 1 whenever the leisure binds. Our numerical examples show that when the discount rate is sufficiently close to or exceeds the interest rate, one may have to investigate unconventional sequences of optimal arcs. Although under benchmark parameterization ignoring the leisure constraint is generally innocuous, there are a number of reasonable parameter ranges where doing so results in serious errors in solutions, yet there is a lack of attention to this issue in the literature.

References

Angeletos, G.-M., Laibson, D., Repetto, A., Tobacman, J., Weinberg, S. 2001. The Hyperbolic Consumption Model: Calibration, Simulation, and Empirical Evaluation. *Journal of Economic Perspectives* 15(3), 47-68.

Auerbach, A., Kotlikoff, L., Skinner, J. 1981. The Efficiency Gains from Dynamic Tax Reform. NBER working paper No. 819.

Bagchi, S. 2011. Demographic Shocks, Retirement Decisions and Optimal Social Security. Doctoral thesis, Utah State University.

Bagchi, S., Feigenbaum, J. 2013a. Optimal Income-Tax Progressivity in an Overlapping-Generations Model. Working paper, Utah State University.

Bagchi, S., Feigenbaum, J. 2013b. Is Smoking a Fiscal Good? *Review of Economic Dynamics*, in press.

Barucci, E., Marazzina, D. 2008. Optimal Investment, Stochastic Labor Income and Retirement. *Applied Mathematics and Computation* 218(9), 5588-5604.

Betts, J. 2010. *Practical Methods for Optimal Control and Estimation using Nonlinear Programming.* SIAM, USA.

Bullard, J., Feigenbaum, J. 2007. A Leisurely Reading of the Life-cycle Consumption Data. *Journal of Monetary Economics* 54(8), 2305-2320.

Bütler, M., 2001. Neoclassical Life-cycle Consumption: A Textbook Example. *Economic Theory* 17(1), 209-221.

Caliendo, F., 2011. Time-inconsistent Preferences and Social Security: Revisited in Continuous Time. *Journal of Economic Dynamics and Control* 35(5), 668-675.

Caliendo, F., Gahramanov, E. 2013. Myopia and Pensions in General Equilibrium. *Journal of Economics and Finance* 37(3), 375-401.

Chachuat, B. 2007. *Nonlinear and Dynamic Optimization: from Theory to Practice.* Automatic Control Laboratory, EPFL, Switzerland.

Choi, K., Shim, G., Shin, Y. 2008. Optimal Portfolio, Consumption-Leisure and Retirement Choice Problem with CES Utility. *Mathematical Finance* 18(3), 445-472.

Conesa, J., Garriga, C. 2008. Optimal Fiscal Policy in the Design of Social Security Reforms. *International Economic Review* 49(1), 291-318.

Cremer, H., Lozachmeur, J-M., Pestieau, P. 2004. Social Security, Retirement Age and Optimal Income Taxation. *Journal of Public Economics* 88(11), 2259-2281.

Davies, J. 1981. Uncertain Lifetime, Consumption, and Dissaving in Retirement. *Journal of Political Economy* 89(3), 561-577.

Farhi, E., Panageas, S. 2007. Saving and Investing for Early Retirement: A Theoretical Analysis. *Journal of Financial Economics* 83 (1), 87-121.

Feigenbaum, J. 2008. Can Mortality Risk Explain the Consumption Hump? *Journal of Macroeconomics* 30(3), 844-72.

Fuster, L., İmrohoroğlu, A., İmrohoroğlu, S. 2003. A Welfare Analysis of Social Security in a Dynastic Framework. *International Economic Review* 44(4), 1247-1274.

Gahramanov, E., Tang, X. 2013. Should We Refinance Unfunded Social Security? *Economica* 80(319), 532-565.

Gourinchas, P., Parker, J. 2002. Consumption Over the Life Cycle. *Econometrica* 70(1), 47-89.

Gregory, J., Lin, C. 1992. *Constrained Optimization in the Calculus of Variations and Optimal Control Theory.* Chapman & Hall, London.

Gruber, J., Wise D. 2002. Social Security Programs and Retirement Around the World: Micro Estimation, NBER working paper No. 9407.

Gruber, J., Wise D. 2005. Social Security Programs and Retirement Around the World: Fiscal Implications, NBER working paper No. 11290.

Gustman, A., Steinmeier, T. 2005. The Social Security Early Entitlement Age in a Structural Model of Retirement and Wealth. *Journal of Public Economics* 89(2-3), 441-463.

Hakansson, N. 1969. Optimal Investment and Consumption Strategies Under Risk, an Uncertain Lifetime, and Insurance. *International Economic Review* 10(3), 443-466.

Hansen, G., İmrohoroğlu, S. 2008. Consumption Over the Life Cycle: The Role of Annuities. *Review of Economic Dynamics* 11(3), 566-583.

Heckman, J. 1974. Life Cycle Consumption and Labor Supply: An Explanation of the Relationship between Income and Consumption Over the Life Cycle. *American Economic Review* 64(1), 188-194.

Heckman, J., Macurdy, T. 1980. A Life Cycle Model of Female Labour Supply. *Review of Economic Studies* 47(1), 47-74.

Heijdra, B., Ligthart, J. 2002. Tax Policy, the Macroeconomy, and Intergenerational Distribution. *IMF Staff Papers* 49(1), 106-127.

Huang, H., Milevsky, M., Salisbury, T. 2012. Optimal Retirement Consumption with a Stochastic Force of Mortality. *Insurance: Mathematics and Economics* 51(2), 282-291.

İmrohoroğlu, A., İmrohoroğlu, S., Joines, D. 1995. A Life Cycle Analysis of Social Security. *Economic Theory* 6(1), 83-114.

Kamien, M., Schwartz, N. 1981. *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management.* Elsevier North Holland, New York.

Koo, J., Koo, B., Shin, Y. 2013. An Optimal Investment, Consumption, Leisure, and Voluntary Retirement Problem with Cobb-Douglas Utility: Dynamic Programming Approaches. *Applied Mathematics Letters* 26(4), 481-486.

Ladrón-de-Guevara, A., Ortigueira, S., Santos, M., 1999. A Two-Sector Model of Endogenous Growth with Leisure. *Review of Economic Studies* 66(3), 609-631.

Lenhart, S., Workman, J., 2007. *Optimal Control Applied to Biological Models*. Chapman & Hall/CRC.

Park, H. 2011. A Lifecycle Model of Charitable Giving: A Quantitative Analysis of Social Preferences, University of Pittsburgh working paper.

Prescott, E., Rogerson, R., Wallenius, J. 2009. Lifetime Aggregate Labor Supply with Endogenous Workweek Length. *Review of Economic Dynamics* 12(1), 23-36.

Rogerson, R. 2011. Individual and Aggregate Labor Supply with Coordinated Working Times. *Journal of Money, Credit and Banking*, Supplement to 43(5), 7-37.

Rogerson, R., Wallenius, J. 2009. Micro and Macro Elasticities in a Life Cycle Model with Taxes. *Journal of Economic Theory* 144(6), 2277-2292.

Ross, M. 2009. *A Primer on Pontryagin's Principle in Optimal Control*. Collegiate Publishers, San Francisco.

Samwick, A. 1998. New Evidence on Pensions, Social Security, and the Timing of Retirement. *Journal of Public Economics* 70(2), 207-236.

Skinner, J. 1985. Variable Lifespan and the Intertemporal Elasticity of Consumption. *Review of Economics and Statistics* 67(4), 616-623.

Vidangos, I., 2009. Household Welfare, Precautionary Saving, and Social Insurance under Multiple Sources of Risk, Federal Reserve Board working paper.

von Stryk, O., 1993. 'Numerical Solution of Optimal Control Problems by Direct Collocation'. In: Bulirsch, R., Miele, A., Stoer, J., Well, K.-H. (Eds.), *Optimal Control - Calculus of Variations, Optimal Control Theory and Numerical Methods*, International Series of Numerical Mathematics, vol. 111. Basel, Birkhäuser, 129-143.

Yaari, M. 1965. Uncertain Lifetime, Life Insurance, and the Theory of the Consumer. *Review of Economic Studies* 32(2), 137-150.

Zeldes, S. 1989. Optimal Consumption with Stochastic Income: Deviations from Certainty Equivalence. *Quarterly Journal of Economics* 104(2), 275-298.

Zhang, A. 2010. A Closed-form Solution for the Continuous-time Consumption Model with Endogenous Labor Income. *Decisions in Economics and Finance* 33(2), 149-167.